J. Fluid Mech. (1988), *vol.* 195, *pp.* 581–586
Printed in Great Britain

Drag of a corrugated plate in a rotating fluid

By C. Y. WANG

Departments of Mathematics and Mechanical Engineering, Michigan State University, East Lansing, MI 48824, USA

(Received 1 December 1987 and in revised form 11 March 1988)

A corrugated plate is translating in a rotating fluid. Assuming low Reynolds number and small amplitudes compared to the Ekman thickness, a perturbation solution is found to second order. The resistance and power due to drag depend on the relative orientation of the corrugations with the motion. In certain instances, it is easier to move a corrugated plate than a flat plate in a rotating fluid.

1. Introduction and formulation

Consider a wavy or corrugated plate which is rotating at the angular velocity of a viscous fluid, and at the same time translating laterally. If the plate were flat, the well-known Ekman boundary layer describes the flow (see e.g. Batchelor 1967). The present paper studies the effects of small-amplitude waviness of the plate on the Ekman solution. The results are important in the drag determination of corrugated or striated surfaces moving in a rotating fluid. An applicable example is the study of the resistance of large platforms in the Arctic.

Let the Cartesian system (x', y', z') be rotating with angular velocity Ω in the z'-direction. Let (u', v', w') be the respective velocity components. Figure 1 shows the plate is moving, in its own neutral plane with velocity U which is at an angle β with the direction of the corrugations. Let the plate be described by

$$z' = h \cos\left[\frac{2\pi}{l}(x' - U \sin\beta t\right]. \tag{1}$$

Here h is the amplitude and l is the wavelength of the corrugations. The velocity components on the plate are

$$u' = U \sin\beta, \quad v' = U\cos\beta, \quad w' = 0. \tag{2}$$

As $z' \rightarrow \infty$, all velocities are zero. The solution will be independent of the y'-direction. The time variable t can be eliminated by translating the coordinate system with speed $U \sin\beta$ in the x'-direction. Set

$$u' = U\sin\beta + Uu, \quad v' = Uv, \quad w' = Uw, \tag{3}$$

$$x' = U\sin\beta t + x(\nu/\Omega)^{\frac{1}{2}}, \quad y' = y(\nu/\Omega)^{\frac{1}{2}}, \quad z' = z(\nu/\Omega)^{\frac{1}{2}}. \tag{4}$$

Here ν is the kinematic viscosity and we recognize $(\nu/\Omega)^{\frac{1}{2}}$ as the Ekman thickness. The Navier–Stokes equations in the rotating (x', y', z') system become

$$-2v + a\epsilon[(\sin\beta + u)\,u_x + wu_z] = -p_x + u_{xx} + u_{zz}, \tag{5}$$

$$2(\sin\beta + u) + a\epsilon[(\sin\beta + u)\,v_x + wv_z] = v_{xx} + v_{zz}, \tag{6}$$

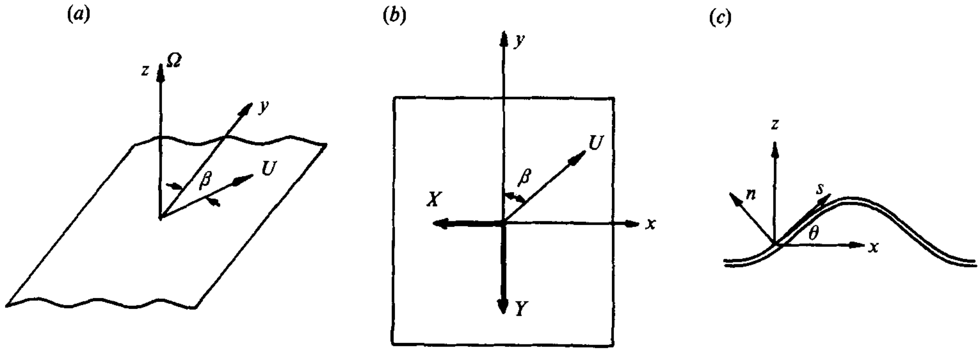

FIGURE 1. The coordinate systems.

$$a\epsilon[(\sin\beta+u)w_x+ww_z] = -p_z+w_{xx}+w_{zz}, \tag{7}$$

$$u_x+w_z = 0. \tag{8}$$

Here we have assumed small-amplitude corrugations and low velocity such that $h/(\nu/\Omega)^{\frac{1}{2}} \equiv \epsilon \ll 1$ and $U/(\nu\Omega)^{\frac{1}{2}} \equiv a\epsilon \ll 1$ where a is a constant of order unity. The boundary conditions are on $z = \epsilon\cos(2\pi(\nu/\Omega)^{\frac{1}{2}}x/l) \equiv \epsilon\cos\lambda x$

$$u = 0, \quad v = \cos\beta, \quad w = 0, \tag{9}$$

and far from the surface at $z\rightarrow\infty$,

$$u = -\sin\beta, \quad v = 0, \quad w = 0. \tag{10}$$

2. Perturbation solution

We expand

$$u = u_0+\epsilon u_1+\epsilon^2 u_2+\dots \tag{11}$$

and similarly for u, w, p. The boundary condition (9) is expanded in a Taylor series about $z = 0$. Without going into the details, (5)–(10) yield the zeroth-order solution which is similar to the Ekman solution

$$u_0 = -\sin\beta+\sin(\beta+z)\,e^{-z}, \quad v_0 = \cos(\beta+z)\,e^{-z}, \quad w_0 = 0, \quad p_0 = \text{constant}. \tag{12}$$

The first-order solution is periodic in x:

$$\left.\begin{aligned} u_1 &= \cos\lambda x f'(z), & v_1 &= \cos\lambda x g(z), \\ w_1 &= \lambda\sin\lambda x f(z), & p_1 &= \lambda\sin\lambda x h(z), \end{aligned}\right\} \tag{13}$$

where

$$f = c_1\,e^{\gamma_1 z}+c_2\,e^{\gamma_2 z}+c_3\,e^{\gamma_3 z}, \tag{14}$$

and $\gamma_1, \gamma_2, \gamma_3$ are the three roots of

$$\gamma^6-3\lambda^2\gamma^4+(3\lambda^4+4)\gamma^2-\lambda^6 = 0. \tag{15}$$

Figure 2 shows the three computed roots as a function of λ. An analytic solution of (15) is possible when $\lambda = 2$.

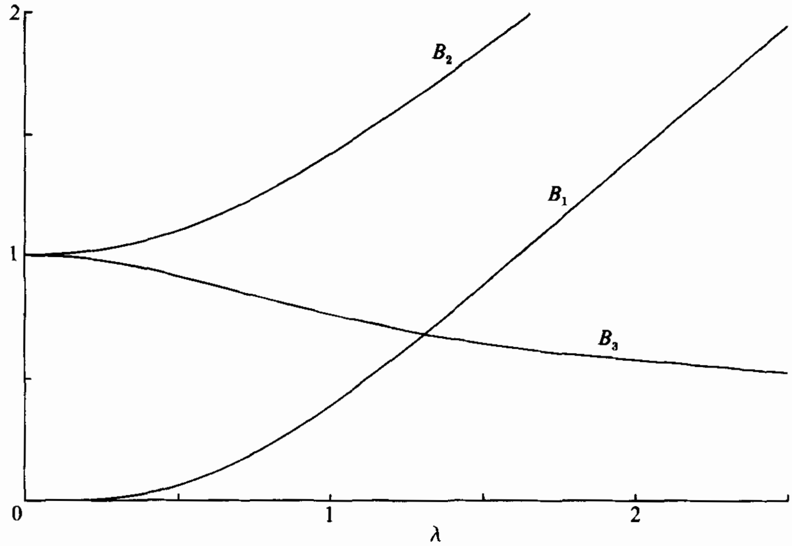

FIGURE 2. The roots of equation (15) $\gamma_1 = -B_1, \gamma_2 = -B_2 + B_3\,\mathrm{i}, \gamma_3 = -B_2 - B_3\,\mathrm{i}$.

The boundary conditions dictate

$$\left.\begin{aligned} c_3 &= \frac{K(\gamma_2-\gamma_1)\,\mathrm{i} - I(\sin\beta - \cos\beta)\,\mathrm{i}}{2\,\mathrm{Im}\,[(\gamma_3-\gamma_1)\,I]}, \\ K &\equiv -2\lambda^2(\sin\beta+\cos\beta) - (\lambda^4+4)\,(\sin\beta - \cos\beta), \\ I &\equiv \gamma_2^5 - 2\lambda^2\gamma_2^3 - \gamma_1^5 + 2\lambda^2\gamma_1^3, \\ c_2 &= \text{complex conj. } c_3, \quad c_1 = -2\,\mathrm{Re}\,c_3. \end{aligned}\right\} \tag{16}$$

The functions g and h are related to f by

$$g = \frac{-1}{2\lambda^2}[f^v - 2\lambda^2 f''' + (\lambda^4+4)f'], \tag{17}$$

$$h = \frac{2g}{\lambda^2} - f' + \frac{f'''}{\lambda^2}. \tag{18}$$

The solutions are plotted in figure 3 for $\beta = 0, \frac{1}{2}\pi$ and $\lambda = 1$. All functions show oscillatory decay within the Ekman layer.

In the second-order solutions, we are only interested in the non x-periodic terms since these give rise to mean drag. Let an overbar denote the integration in x over the period $2\pi/\lambda$. The governing equations yield

$$\begin{gathered} -2\overline{v_2} = \overline{u_{2zz}}, \quad 2\overline{u_2} = \overline{v_{2zz}}, \\ 0 = -\overline{p_{2z}} + \overline{w_{2zz}}, \quad \overline{w_{2z}} = 0. \end{gathered} \tag{19}$$

The boundary conditions are

$$\overline{u_2}|_0 = -\overline{\cos\lambda x u_{1z}}|_0 - \tfrac{1}{2}\overline{(\cos^2\lambda x)\,u_{0zz}}|_0, \quad \overline{u_2}|_\infty = 0, \tag{20}$$

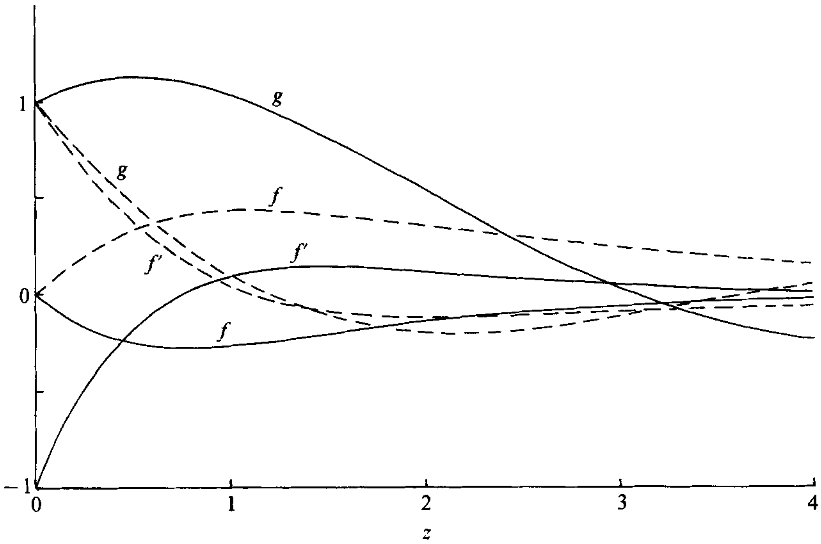

FIGURE 3. The functions $f(x), g(z), f'(z)$ for $\lambda = 1$. ——, $\beta = 0°$; – – –, $\beta = 90°$.

with similar conditions for $\bar{v}_2$ and $\bar{w}_2$. The solution is

$$\left.\begin{aligned}\bar{u}_2 &= \tfrac{1}{2}\,\mathrm{e}^{-z}\{[\cos\beta - f''(0)]\cos z - [\sin\beta + g'(0)]\sin z\},\\ \bar{v}_2 &= \tfrac{1}{2}\,\mathrm{e}^{-z}\{[f''(0) - \cos\beta]\sin z - [\sin\beta + g'(0)]\cos z\},\\ \bar{w}_2 &= 0, \quad \bar{p}_2 = 0.\end{aligned}\right\} \tag{21}$$

3. Stresses and drag on the plate

Let s be the direction along the surface of the plate and n be the normal (figure 1c). The stresses on the plate in the intrinsic directions are

$$\begin{aligned}\tau_{ny} = {}& \frac{\epsilon\lambda\sin\lambda x}{(1+\epsilon^2\lambda^2\sin^2\lambda x)^{\frac{1}{2}}}(v_x|_0 + \epsilon\cos\lambda x v_{xz}|_0)\\ &+ \frac{1}{(1+\epsilon^2\lambda^2\sin^2\lambda x)^{\frac{1}{2}}}(v_z|_0 + \epsilon\cos\lambda x v_{zz}|_0 + \tfrac{1}{2}\epsilon^2\cos^2\lambda x v_{zzz}|_0) + O(\epsilon^3),\end{aligned} \tag{22}$$

$$\begin{aligned}\tau_{nn} = {}& \frac{2\epsilon\lambda\sin\lambda x}{1+\epsilon^2\lambda^2\sin^2\lambda x}[w_x|_0 + u_z|_0] + \frac{1}{1+\epsilon^2\lambda^2\sin^2\lambda x}[-p|_0 + 2w_z|_0\\ &+ \epsilon\cos\lambda x(-p_z|_0 + 2w_{zz}|_0)] + O(\epsilon^2),\end{aligned} \tag{23}$$

$$\begin{aligned}\tau_{ns} = {}& \frac{\epsilon\lambda\sin\lambda x}{1+\epsilon^2\lambda^2\sin^2\lambda x}[-p|_0 + 2u_x|_0 + \epsilon\cos\lambda x(-p_z|_0 + 2u_{xz}|_0)]\\ &+ \frac{1-\epsilon^2\lambda^2\sin^2\lambda x}{1+\epsilon^2\lambda^2\sin^2\lambda x}[w_x|_0 + u_z|_0 + \epsilon\cos\lambda x(w_{xz}|_0 + u_{zz}|_0)\\ &+ \tfrac{1}{2}\epsilon^2\cos^2\lambda x(w_{xzz}|_0 + u_{zzz}|_0)]\\ &- \frac{\epsilon\lambda\sin\lambda x}{1+\epsilon^2\lambda^2\sin^2\lambda x}[-p|_0 + 2w_z|_0 + \epsilon\cos\lambda x(-p_z|_0 + 2w_{zz}|_0)] + O(\epsilon^3).\end{aligned} \tag{24}$$

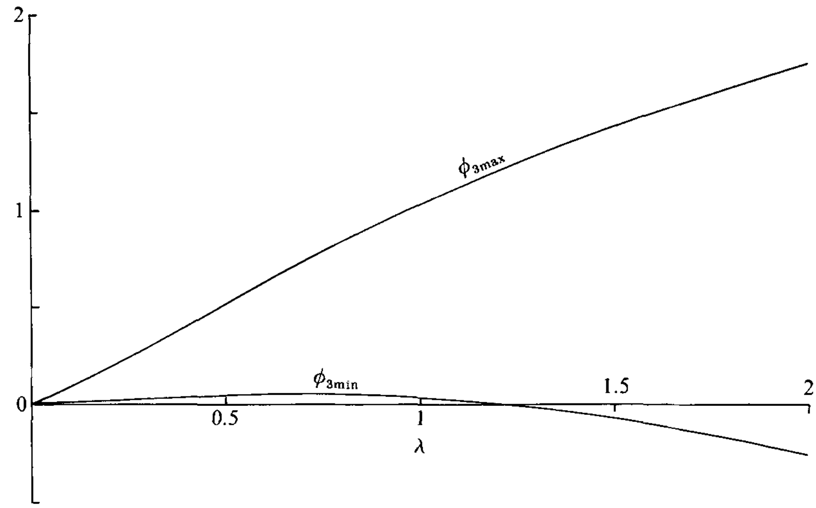

FIGURE 4. The extremum values of ϕ_3 which is related to power.

The drag per unit width per period in the $-y$-direction is solely due to shear.

$$Y = -\int \tau_{ny}\, \mathrm{d}s = \oint \tau_{ny}(1+\epsilon^2\lambda^2 \sin \lambda x)^{\frac{1}{2}}\, \mathrm{d}x$$

$$= \sin\beta + \cos\beta + \epsilon^2\phi_1 + O(\epsilon^4) \tag{25}$$

$$\phi_1 \equiv -\tfrac{1}{2}g'(0) - \tfrac{1}{2}f''(0) - f'(0). \tag{26}$$

The drag per width per period in the $-x$-direction is the sum of shear drag and pressure drag.

$$X = -\oint \tau_{ns}\, \mathrm{d}x - \oint \tau_{nn}\, \epsilon\lambda \sin \lambda x\, \mathrm{d}x$$

$$= \sin\beta - \cos\beta + \epsilon^2\phi_2 + O(\epsilon^4), \tag{27}$$

$$\phi_2 \equiv g(0) - \tfrac{1}{2}f''(0) + \tfrac{1}{2}g'(0). \tag{28}$$

If the plate were flat, $\epsilon = 0$. Equations (25), (27) show the net resistance (per projected area, normalized by $\rho U(\nu\Omega)^{\frac{1}{2}}$) has the value of $\sqrt{2}$ pointing at an angle of 135° from the direction of motion. This non-alignment is characteristic of drag in a rotating fluid. If $\epsilon \neq 0$, the quantities $\epsilon^2\phi_1$ and $\epsilon^2\phi_2$ represent corrections to the resistance components in the $-y$- and $-x$-directions. Of interest is the power P (per area, normalized by $\rho U^2(\nu\Omega)^{\frac{1}{2}}$) needed to maintain the motion

$$P = \sin\beta X + \cos\beta Y = 1 + \epsilon^2\phi_3 + O(\epsilon^4), \tag{29}$$

$$\phi_3 \equiv \sin\beta\phi_2 + \cos\beta\phi_1. \tag{30}$$

Figure 4 shows the maximum of ϕ_3, occurring at $\beta = 90°$ increases with λ. In situations where the plate needs to be slowed down, maximum power dissipation is desirable. The minimum power, occurring at $\beta = 0°, 180°$, is particularly important. Figure 4 shows ϕ_3 min is slightly positive for $0 < \lambda < 1.171$, but becomes negative for $\lambda > 1.171$. This means, in a rotating fluid, it may be easier to move a corrugated plate than a flat plate!

4. Discussion

Wang (1978) studied the low-Reynolds-number flow due to a moving corrugated plate in a non-rotating fluid. It was found that the corrugations always increase the drag. There is also a directional preference. The drag increase when the motion is against the striations is about twice the drag increase when motion is along the striations. The situation is quite different when the fluid is rotating. The present paper shows the power (or drag in the direction of motion) for the corrugated plate may be less than that of a flat plate. The reason is as follows. For any parallel motion in the plane of rotation an Ekman flow is induced 90° from the direction of motion. Thus when the plate is moving in the y-direction ($\beta = 0$), a flow u_0 is induced. In order to bring the velocity to zero at the crests of the corrugations, a negative u_1 (and a positive $u_{1z}|_0$) is needed. This in turn causes a negative compensatory $\overline{u_2|_0}$ for both crests and valleys, represented by the term $-\overline{\cos \lambda x u_{1z}|_0}$ in equation (20). Since $\overline{u_2}$ is in the $-x$-direction, an Ekman flow $\overline{v_2}$ in the y-direction is induced. This $\overline{v_2}$ gives a positive shear $f''(0)$ which reduces the drag Y and thus the power. Physically the corrugations restrict the Ekman side flow, resulting in a saving in energy.

The assumption of small $U/(\nu\Omega)^{\frac{1}{2}}$, (Rossby number × Reynolds number)$^{\frac{1}{2}}$, implies small inertial effects in comparison to viscous and rotational effects. For the present problem, the nonlinear convective terms are suppressed until the second order where they have zero mean in x. Thus the reduction of power is not due to inertial convection, but to the nonlinear interaction of the Ekman layer and corrugations described earlier. If $U/(\nu\Omega)^{\frac{1}{2}}$ is not small, the zeroth-order solution is the same while the first order would be a solution to the Orr–Sommerfeld equation. The second order, still containing the nonlinear interactions, would be unsolvable.

The amplitude of the corrugations is assumed to be small compared to the Ekman thickness. If the amplitude is the same order as the Ekman thickness, the Ekman layer is destroyed and an analytic solution does not exist. However, if the Ekman thickness is much smaller than the amplitude, then we can envisage a thin Ekman boundary layer attached to the surface contour. Locally, the boundary is like a slanted flat surface. The zeroth-order solution to this problem was briefly described by Hsueh (1968). Changes in the mean drag, of course, cannot be obtained from the zeroth-order.

REFERENCES

BATCHELOR, G. K. 1967 *An Introduction to Fluid Dynamics*, chapter 4. Cambridge University Press.

HSUEH, Y. 1968 Viscous fluid flow over a corrugated bottom in a strongly rotating system. *Phys. Fluids* **11**, 940–944.

WANG, C. Y. 1978 Drag due to a striated boundary in slow Couette flow. *Phys. Fluids* **21**, 697–698.